# Iterative Explicit Guidance for Low Thrust Spaceflight

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**A retargeting procedure is developed for use as a nonlinear low-thrust guidance scheme. The selection of a control program composed of a sequence of inertially fixed thrust-acceleration vectors permits all trajectory computations to be made with closed-form expressions, and allows the controls to be represented by constant parameters, thrust-acceleration vectors, and thrusting times. By requiring each trajectory to be time optimal, the guidance problem is transformed into a parameter optimization problem which is solved by a Davidon-type method. The scheme is applied to a low-thrust capture mission, and the results of computer simulations are presented.**

## I. Introduction

IN the near future, the capability for unmanned exploration of the solar system will be expanded by the introduction of low-thrust interplanetary spaceflight. To date, proposed methods of guidance for these vehicles have generally been based on linear perturbation theory; namely, the spacecraft's equations of motion are linearized with respect to a numerically generated reference trajectory and a guidance scheme is designed to control the resultant linear system. References 1–6 describe a number of these schemes.

Most linear guidance laws continuously modify a continuously varying reference control program and, consequently, lack the simplicity desired for reliability and ease of implementation. In addition, they are of limited flexibility because of their dependence on a reference trajectory, and they may lead to control divergence if perturbations outside the linearity region are encountered. In an effort to overcome the limitations of linear guidance, nonlinear or explicit schemes have been developed. Explicit guidance is basically a retargeting procedure and therefore is more computationally complex. Early practical schemes[7,8] depended upon approximations and simplifying assumptions to obtain computational tractability. However, improvements in computer technology and computational methods have reduced the need for such simplification and have led to the proposal of more sophisticated explicit methods.[9–11]

In this paper an explicit low-thrust guidance scheme is presented which can be used in regions of space where local gravitational accelerations are of the same order of magnitude as the vehicle's thrust acceleration. A reduction in computational effort is realized by taking the control program to be a sequence of inertially fixed constant thrust-acceleration vectors for specified time periods. Under that assumption, a closed-form solution for the motion of a spacecraft in a central force field can be obtained and numerical prediction of the low-thrust trajectory is avoided. Moreover, the representation of the control program by constants permits the use of sophisticated parameter optimization techniques in obtaining control updates. In the following sections, the closed-form trajectory solution is discussed, and the nonlinear guidance scheme employing it and a parameter optimization method are presented and evaluated. Although only planar flight is considered, all results generalize to the three-dimensional case.

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## II. The Constant Thrust-Acceleration Closed-Form Solution

Beletskii[12] has developed a solution for the three-dimensional motion of a vehicle with a constant thrust-acceleration vector in an inverse square gravitational field. The equations of planar motion for such a vehicle may be written as

$$\ddot{r} - r\dot{\theta}^2 = -1/r^2 + \varepsilon\cos(\alpha-\theta) \tag{1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \varepsilon\sin(\alpha-\theta) \tag{2}$$

where $r$ = radial distance, $\theta$ = polar angle, $\varepsilon$ = constant thrust-acceleration magnitude, $\alpha$ = constant thrust-acceleration direction angle, $(\dot{\ })$ = first-time derivative, and $(\ddot{\ })$ = second-time derivative. The variables in Eqs. (1) and (2) have been nondimensionalized with respect to conditions on a circular orbit of arbitrary radius.

The two integrals for Eqs. (1) and (2) that may be obtained from work-energy and moment of momentum considerations are

$$\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - 1/r - \varepsilon r\cos(\theta-\alpha) = E \tag{3}$$

$$r^2\dot{r}\dot{\theta}\sin(\theta-\alpha) + (r^3\dot{\theta}^2 - 1)\cos(\theta-\alpha) + \tfrac{1}{2}\varepsilon r^2\sin^2(\theta-\alpha) = c \ . \tag{4}$$

By the introduction of the new dependent and independent variables

$$u = r[1-\cos(\theta-\alpha)], \qquad v = r[1+\cos(\theta-\alpha)] \tag{5}$$

$$\tau = \int_{t_o}^{t} dt/r \tag{6}$$

where $t$ = nondimensional time and $t_o$ = initial value of $t$, integrals (3) and (4) may be written as

$$(du/d\tau)^2 = 2(1+c)u + 2Eu^2 - \varepsilon u^3 \tag{7}$$

$$(dv/d\tau)^2 = 2(1-c)v + 2Ev^2 + \varepsilon v^3 \tag{8}$$

which can be put into standard elliptic form

$$\pm(\varepsilon)^{1/2}\tau = \textstyle\int_{u_o}^{u} du/[u(u_1-u)(u-u_2)]^{1/2} \tag{9}$$

$$\pm(\varepsilon)^{1/2}\tau = \textstyle\int_{v_o}^{v} dv/[v(v_1-v)(v_2-v)]^{1/2} \tag{10}$$

with

$$u_1, u_2 = (1/\varepsilon)\{E \pm [E^2 + 2\varepsilon(1+c)]^{1/2}\}, \qquad u_1 > u_2$$

$$v_1, v_2 = (1/\varepsilon)\{E \pm [E^2 - 2\varepsilon(1-c)]^{1/2}\}, \qquad v_1 > v_2$$

$$u_o, v_o = \text{values of } u, v \text{ at } t_o$$

The signs for Eqs. (9) and (10) are defined by the initial conditions and the relations

$$du/d\tau = \pm[\varepsilon u(u_1-u)(u-u_2)]^{1/2} = r\dot{r}[1-\cos(\theta-\alpha)]+r^2\dot{\theta}\sin(\theta-\alpha) \quad (11)$$

$$dv/d\tau = \pm[\varepsilon v(v_1-v)(v_2-v)]^{1/2} = r\dot{r}[1+\cos(\theta-\alpha)]-r^2\dot{\theta}\sin(\theta-\alpha) \quad (12)$$

Numerous solutions for Eqs. (9) and (10) exist depending upon initial conditions, and all of those having physical significance are discussed by Beletskii. As examples, two of the solutions are presented below. From Eq. (9) with $u_2 < 0 \leqq u \leqq u_1$

$$u = u_1 \operatorname{cn}^2 w \quad (13)$$

where

$$w = \mp\tfrac{1}{2}[\varepsilon(u_1-u_2)]^{1/2}\tau + F(\phi_o, k)$$

$$\sin\phi_o = [(u_1-u_o)/u_1]^{1/2}, \qquad k = [u_1/(u_1-u_2)]^{1/2}$$

From Eq. (10) with $0 < v$, $v_1$, $v_2$ complex

$$v = A(1-\operatorname{cn} w)/(1+\operatorname{cn} w) \quad (14)$$

where $w = \pm(\varepsilon A)^{1/2}\tau + F(\phi_o, k)$

$$\cos\phi_o = (A-v_o)/(A+v_o)$$

$$k = [(A+b)/2A]^{1/2}$$

$$A = [2(1-c)/\varepsilon]^{1/2}, \qquad b = -E/\varepsilon$$

Solutions for the other cases are similar to Eqs. (13) and (14) and are presented in Ref. 12.

To complete the description of a trajectory, the following transformation between $u$, $v$, and physical variables is required

$$r = \tfrac{1}{2}(u+v) \quad (15)$$

$$dr/dt = \frac{1}{2r}(du/d\tau + dv/d\tau) \quad (16)$$

$$\cos(\theta-\alpha) = \frac{1}{2r}(v-u) \quad (17)$$

$$d/dt[\cos(\theta-\alpha)] = \tfrac{1}{2}r^3\cdot(u\,dv/d\tau - v\,du/d\tau) \quad (18)$$

$$t - t_o = \int_o^\tau \tfrac{1}{2}(u+v)\,d\tau \quad (19)$$

Since $\dot{\theta}$ normally does not change sign, the quadrant of $\theta$ can be found by using

$$\operatorname{sign}[\sin(\theta-\alpha)] = \operatorname{sign}[v\,du/d\tau - u\,dv/d\tau] \quad (20)$$

which was obtained from Eq. (18). The integral in Eq. (19) may also be written in closed form. As an example, for solutions (13) and (14) the indefinite integrals

$$\int \operatorname{cn}^2 w\,dw = (1/k^2)[E(\phi,k)-(1-k^2)w] \quad (21)$$

$$\int[(1-\operatorname{cn} w)/(1+\operatorname{cn} w)]\,dw = w - 2E(\phi,k) - 2(\operatorname{sn} w \operatorname{cn} w)/(1+\operatorname{cn} w) \quad (22)$$

with $\operatorname{tn} w = \tan\phi$ can be employed to compute $t$.

By means of the various expressions previously derived, it is possible to analytically describe the motion of a vehicle with a constant thrust-acceleration vector. Given a set of initial conditions, $E$ and $c$ can be evaluated, and the correct cases and sign conventions identified. Then for a specified time $t$, $\tau$ can be found from Eq. (19), $u$ and $v$ computed, and the state determined from Eqs. (15–18).

## III. Guidance Philosophy

For planar flight, a vehicle employing inertially fixed thrust-acceleration vector control is limited in its ability to attain a desired terminal condition because only three parameters may be selected to define the trajectory, i.e., thrust-acceleration magnitude and direction, and time of powered flight. In order to provide the flexibility necessary for guidance, the admissible class of control programs is therefore extended to include those which may be represented as a sequence of fixed thrust-acceleration vectors. Such programs are described by the set of elements

$$[\varepsilon_i, \alpha_i, \tau_i], \qquad i = 1\text{–}N \quad (23)$$

where $\varepsilon_i$ = thrust acceleration magnitude during $i$th time interval, $\alpha_i$ = thrust direction angle during the $i$th time interval, $\tau_i$ = duration of the $i$th time interval, and $N$ = number of powered flight intervals in the control program. By selection of $N$ and the values of $(\varepsilon_i, \alpha_i, \tau_i)$, a wide variety of trajectories may be generated, and the closed-form expressions of Sec. II may be used to find the vehicle's state at any time.

The objective of the guidance scheme is to find a trajectory defined by a set $[\varepsilon_i, \alpha_i, \tau_i]$ for a given $N$, which attains a desired final state while optimizing some measure of performance. Since the control program is parameterized, the guidance problem is a constrained parameter optimization problem requiring the minimization of

$$J = J_o(\varepsilon, \alpha, \tau) \quad (24)$$

subject to

$$\Delta X_f = 0 \quad (25)$$

where $J_o$ = measure of performance (e.g., flight time), $\Delta X_f$ = vector of terminal state errors, and $\varepsilon, \alpha, \tau$ = $N$-dimensional parameter vectors with components $\varepsilon_i$, $\alpha_i$, $\tau_i$. The most straightforward method for solving the constrained optimization problem is to transform it into an unconstrained one by means of penalty functions; i.e., minimize

$$J = KJ_o + \tfrac{1}{2}\Delta X_f{}^T S \Delta X_f \quad (26)$$

where $K$ = weighting factor and $S$ = weighting matrix. The weights $K$ and $S$ are selected to guarantee satisfaction of Eq. (25) to within an acceptable tolerance.

A number of computational algorithms exist for the solution of parameter optimization problems. For real-time guidance purposes we have selected a Davidon-type method[13] employing numerical partial derivatives. This particular algorithm exhibits good convergence properties for the guidance problem. The use of numerical partials is dictated by the extreme complexity of the analytic partials; however, as Johnson and Kamm[14,15] have shown, numerical differentiation can be employed effectively with accelerated gradient methods. Moreover, because $J$ is evaluated quite accurately by means of closed-form expressions, finite differences yield good approximate derivatives.

In order to generate a new control during a guidance cycle, the Davidon iterator requires an initial guess, and the logical choice is the current control program. This choice has an additional advantage since the first step in the algorithm predicts the trajectory using the initial control estimate. Consequently, if the predicted trajectory minimizes Eq. (26), no control changes are necessary and the update operation can be bypassed. Every time a new control is determined, it will contain a number of elements $(\varepsilon_i, \alpha_i, \tau_i)$ equal to the number unused in the previous control program. Therefore, as the vehicle proceeds to its target, the number of parameters in each updated control decreases, resulting in a loss of controllability which may make satisfaction of condition Eq. (25) difficult. A remedy for this problem is to subdivide the current control in order to introduce additional parameters. For example, if only three parameters

$$(\varepsilon_1, \alpha_1, \tau_1) \quad (27)$$

remain, a six parameter program can be computed using as initial guesses

$$(\varepsilon_1, \alpha_1, \tfrac{1}{2}\tau_1), \quad (\varepsilon_1, \alpha_1, \tfrac{1}{2}\tau_1) \quad (28)$$

The preceding procedure, which can provide six parameters, should be sufficient to define a trajectory with up to four terminal constraints, i.e., a fully constrained planar trajectory. It must be noted, however, that as a spacecraft nears its target point, even the subdivision procedure may not give acceptable results. The reason for this failure is the inherent problem of controllability in low-thrust flight due to the vehicle's inability to perform large maneuvers in short time periods. Consequently, in the final portion of the trajectory when the time-to-go drops below a specified threshold, it is best to terminate guidance and use the current control program to completion.

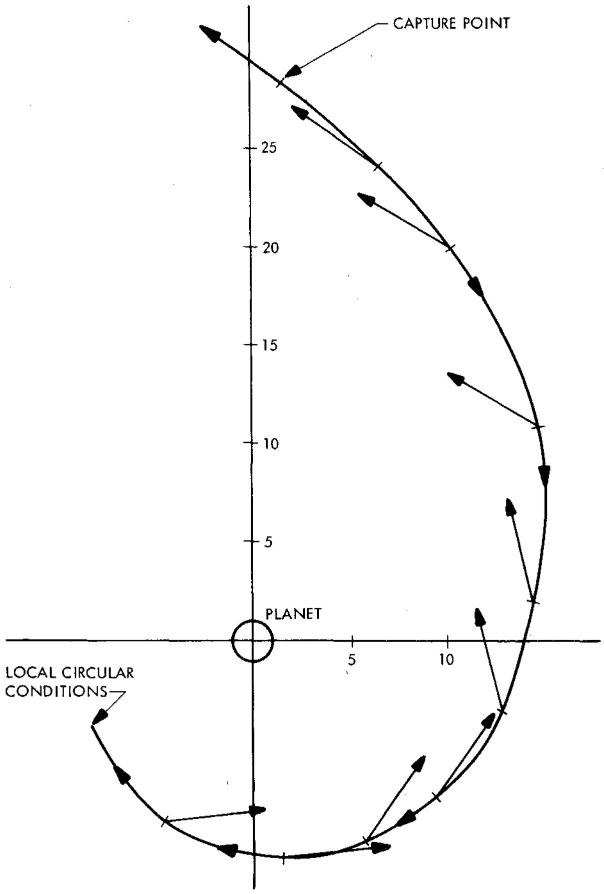


Fig. 1 Nominal trajectory.

## IV. Simulation Results

In order to evaluate the guidance scheme performance, it was applied to the difficult mission of low-thrust planetary approach. The nominal mission profile required the vehicle to proceed from planetary capture to a point near local circular conditions from which a tangential thrust spiral could be used to lower the vehicle to a specified final circular orbit. The gravitational and thrust forces are of the same order of magnitude throughout the flight, and small trajectory perturbations can lead to large terminal errors. The nominal trajectory was obtained from a piecewise constant approximation to the last half revolution of a constant thrust-acceleration tangential thrust spiral trajectory. Figure 1 shows the trajectory, and Tables 1 and 2 give its end conditions and control program.

The guidance scheme was evaluated with respect to its ability to remove initial state errors, and with respect to its performance in a continuously perturbed environment. An initial radial velocity error of $-5\%$ of nominal was assumed at the nominal capture radius, and the trajectory was retargeted from this off-nominal initial state. Figure 2 gives the results of the iterative process; only 30 iterations were required to reduce the terminal errors to an acceptable level (less than 0.01% of the nominal end conditions). Figure 3 shows the initial perturbed trajectory (solid line) and the retargeted one (dotted line). Also shown is the nominal end point. Note that since the mission was to a circular final orbit, the terminal angular position was left uncontrolled, and consequently, the retargeted trajectory does not attain the nominal angular position. In addition to the case of the initial radial velocity error, an initial angular velocity error of 5% was also considered. Although 40 iterations were required to retarget the trajectory, the iteration algorithm performance was essentially the same as in the radial velocity error case.

Testing in a perturbed environment was carried out by simulating an approach during which the vehicle was subjected to random fluctuations in the magnitude and direction of its thrust vector. The standard deviations of these random errors were 2% of nominal for the magnitude and 0.01 rad for the direction; such values are consistent with current thruster noise models.[16] At the end of each period of constant control, the perturbed trajectory was retargeted back to the nominal end conditions. Figure 4 shows the simulated approach for the case of the trajectory with the initial $-5\%$ radial velocity error. The same type of simulation was also performed with the initial angular velocity error trajectory. No difficulty was encountered in either case in successfully correcting the trajectory. Typically, fewer than 16 iterations were required to update the control

Table 1 Nominal trajectory data

| Variable | Initial value | Final value |
|---|---|---|
| Time | 557.30678 | 783.80052 |
| Radial distance | 28.366916 | 9.4429778 |
| Polar angle | 1.6217047 | 5.7923956 |
| Radial rate | −0.14434349 | −0.052113893 |
| Angular rate | 0.0077772959 | 0.034970145 |

Table 2 Nominal control program[a]

| Interval end time | Control angle $\alpha$ (rad) | Interval length parameter $\tau$ |
|---|---|---|
| 582.35533 | 0.59645768 | 0.94084079 |
| 604.66941 | 0.59645582 | 0.94084079 |
| 644.54978 | 0.50922053 | 1.9624139 |
| 676.82545 | 0.50921589 | 1.9624139 |
| 696.97483 | 1.3323747 | 1.4392261 |
| 715.57711 | 1.3323741 | 1.4392261 |
| 730.02235 | 2.1994962 | 1.1941824 |
| 743.71999 | 2.1994955 | 1.1941824 |
| 764.58136 | 3.0516483 | 1.9416263 |
| 783.80052 | 3.0516450 | 1.9416263 |

[a] Thrust acceleration = 0.001 for all intervals.

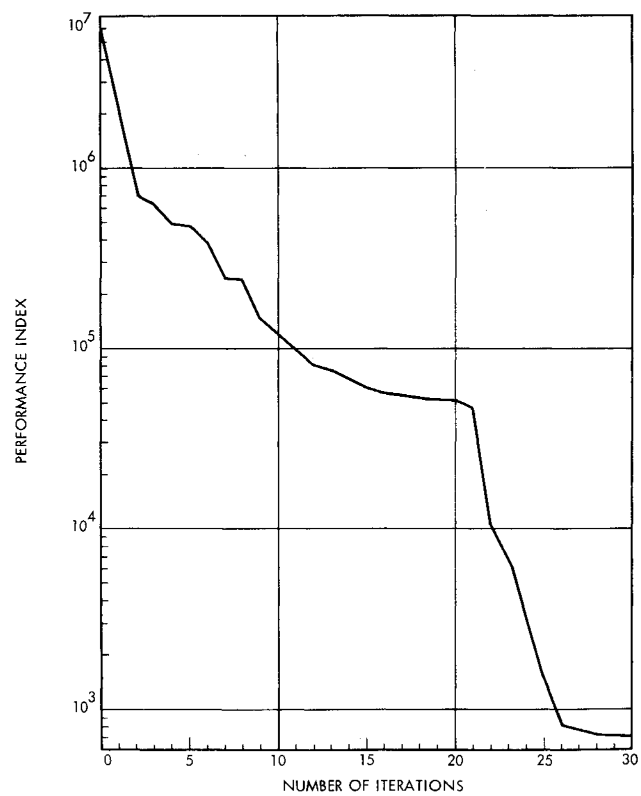


Fig. 2 Removal of initial radial velocity perturbation.

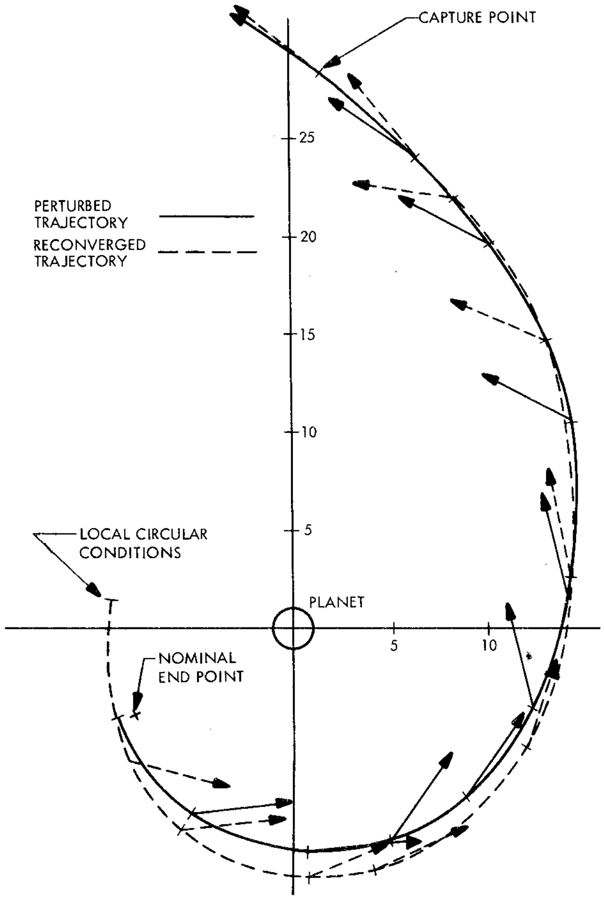


**Fig. 3 Perturbed and reconverged trajectories.**

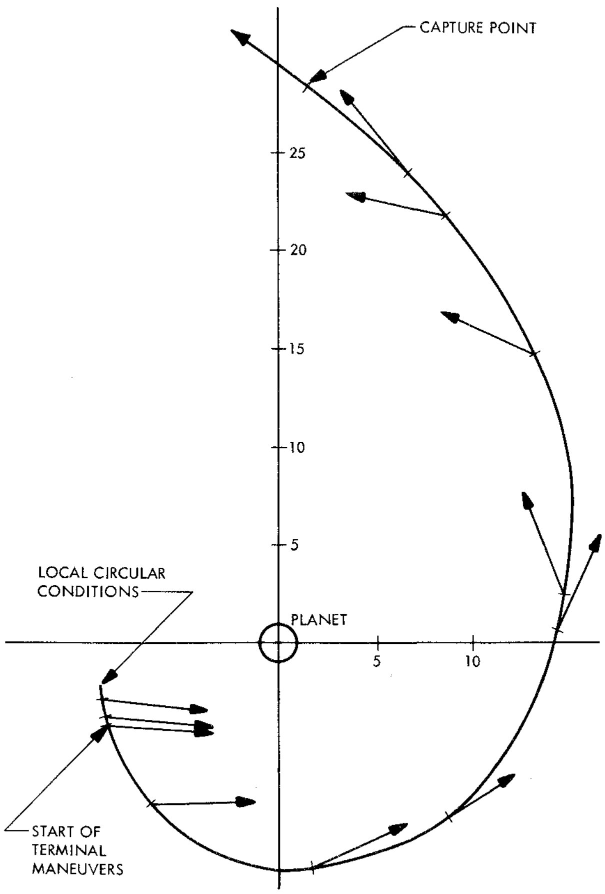


**Fig. 4 Simulated approach trajectory.**

program. Also shown in Fig. 4 is the terminal maneuver strategy of halving the final control interval to enhance controllability. Only two of these "terminal maneuvers" were required to effect successful realization of the specified end conditions.

Since the speed of the iterative process is important for real time guidance, the iteration times in the simulations were noted. When the update involved 20 parameters, each iteration required an average of 2.8 sec on a Univac 1108 computer. When 12 or fewer parameters were sought, the time dropped to less than 1 sec per iteration. Since the thrust vector is nominally held fixed for periods of the order of hours, guidance cycle times of the order of minutes are not unreasonable, and such cycle times are more than adequate for computation of an updated control program.

## V. Conclusions

In this paper, an explicit low-thrust guidance technique which consists of a sequence of inertially fixed constant thrust-acceleration vectors for varying time intervals has been presented. Such a control program permits analytical description of the trajectory and simplicity in control implementation. The control updates are obtained from an optimization procedure which determines the sequence of inertial directions for the thrust vector and the time periods for which those directions are to be maintained. The guidance scheme is probably most useful in cases where gravitational and thrust forces are of the same order of magnitude, for example, during planetary approach near zero energy conditions.

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